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ON THE CORRESPONDENCE PRINCIPLE IN THE PLANE CREEP PROBLEM OF AGEING HOMOGENEOUS MEDIA WITH DEVELOPING SLITS AND CAVITIES*

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The plane creep problem of ageing homogeneous media is considered. The bulk and shear relaxation kernels are assumed to be distinct. Bulk forces, temperature deformations and stresses prescribed on the whole body boundary are the actions. Representations are obtained for the stress, strain, and displacement in terms of the solution for elasticity theory problems for a domain with a fixed boundary and with slits and cavities growing according to a given law.

For a domain with a moving crack it is proved under certain constraints /1/ that the stresses in the creep problem agree with the stresses in the elasticity problem. For a domain with a fixed boundary, necessary and sufficient conditions are obtained /2/ for agreement between the stresses of the creep and elasticity problems. For a constant Poisson's ratio the problem being studied /3/ is investigated in a more general formulation.

A survey of the research devoted to the correspondence principle in the creep theory of ageing media is presented in /4/.

1. Let a homogeneous isotropic linearly-deformable body possessing the properties of ageing and creep occupy a two-dimensional domain $\Omega(\tau) = \Omega_0 \setminus (\bar{\omega}(\tau) \cup \gamma(\tau))$ ($\bar{\omega}$ is the closure of the domain ω). Here $\tau \in [0, t]$ is the time, Ω_0 is a fixed bounded simply-connected domain, and $\omega_i(\tau)$, $\gamma_j(\tau)$ are quasistatic growing (i.e. $\Omega(\tau_1) \subset \Omega(\tau_2)$ for $\tau_1 > \tau_2$) cavities and slits with given laws of growth

$$\omega(\tau) = \bigcup_{i=1}^{N} \omega_i(\tau), \quad \gamma(\tau) = \bigcup_{i=N+1}^{N+J} \gamma_i(\tau)$$

It is assumed that $\omega_i(\tau)$ are simply-connected domains with piecewise-smooth boundaries $\partial\omega_i(\tau)$ /5/, while $\gamma_i(\tau)$ are simple unclosed curves made up of the smooth arcs $\bar{\omega}_i \cap \bar{\omega}_j = \Lambda$, $i \neq j$, $i, j = 1, \ldots, N$, $\gamma_i \cap \gamma_j = \Lambda$, $i \neq j$, $i, j = N+1, \ldots, N+J$ and given parametrization $\mathbf{x}_j(\zeta, \tau)$, for $\zeta \in [0, 1]$, of the curves $\partial\omega_i(\tau), \gamma_j(\tau)$ and piecewise-continuous in τ.

The boundary $\partial\Omega(\tau)$ of the domain $\Omega(\tau)$ consists of the boundary $\partial\Omega_0$, the cavity boundaries $\partial\omega_i(\tau)$ and the edges $\gamma_j^{\pm}(\tau)$ of the slits $\gamma_j(\tau)$. The bulk forces $\mathbf{f} = \{f_i(\mathbf{x}, \tau)\}$ and the temperature $T(\mathbf{x}, \tau)$ are given for $\mathbf{x} \in \Omega(\tau)$, $i = 1, 2$; the surface loads $\mathbf{F} = \{F_i(\mathbf{x}, \tau)\}$ are defined for $\mathbf{x} \in \partial\Omega(\tau)$, and equilibrium conditions are satisfied for all τ.

The equations of the plane creep problem have the form

$$\varepsilon_{ij}(\mathbf{x}, \tau) = 2^{-1}(u_{i,j}(\mathbf{x}, \tau) + u_{j,i}(\mathbf{x}, \tau)), \quad i, j = 1, 2, \quad \mathbf{x} \in \Omega(\tau) \tag{1.1}$$

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$$\int\limits_{\Omega(\tau)} \sigma_{ij}(\mathbf{x},\tau)\,v_{i,j}(\mathbf{x})\,dx - \int\limits_{\Omega(\tau)} f_i(\mathbf{x},\tau)\,v_i(\mathbf{x})\,dx - \int\limits_{\partial\Omega(\tau)} F_i(\mathbf{x},\tau)\,v_i(\mathbf{x})\,dS = 0, \quad \forall \mathbf{v} \in W_2{}^1(\Omega(\tau)) \tag{1.2}$$

$$\varepsilon_{ij}(\mathbf{x},\tau) = \varepsilon_{ij}{}^B(\mathbf{x},\tau) + W^{-1}[\sigma_{ij}(\mathbf{x},\cdot)](\tau) - \delta_{ij}W^{-1}M[\sigma_{ss}(\mathbf{x},\cdot)](\tau), \quad i,j = 1,2, \quad \mathbf{x} \in \Omega(\tau) \tag{1.3}$$

$$\bar{W}[\varphi(\cdot)](\tau) \equiv 2G(\tau)\Big(\varphi(\tau) - \int\limits_0^\tau R_1(x,\xi)\,\varphi(\xi)\,d\xi\Big)$$

$$L[\varphi(\cdot)](\tau) \equiv E^*(\tau)\Big(\varphi(\tau) - \int\limits_0^\tau R_2(x,\xi)\,\varphi(\xi)\,d\xi\Big)$$

$$M \equiv (L - W)(2L + W)^{-1}, \quad \varepsilon_{ij}{}^B(\mathbf{x},\tau) = \delta_{ij}\alpha W^{-1}(I - 2M)\,L[T(\mathbf{x},\cdot)](\tau), \quad I[\varphi(\cdot)](\tau) \equiv \varphi(\tau) \tag{1.4}$$

$$M \equiv {}^1/_3\,(I - WL^{-1}), \ \varepsilon_{ij}{}^B(\mathbf{x},\tau) = \delta_{ij}\alpha T(\mathbf{x},\tau) \tag{1.5}$$

Here ε_{ij}, σ_{ij}, u_i are the Cartesian components of the strain, stress, and displacement, respectively, $G(\tau)$, $R_1(\tau,\xi)$ are the shear modulus and relaxation kernel under shear, $E^*(\tau)$, $R_2(\tau,\xi)$ are the bulk expansion modulus and the bulk relaxation kernel, α is the coefficient of linear expansion, and the operator M and the forced strain $\varepsilon_{ij}{}^B$ are defined by (1.4) for plane strain and (1.5) for the plane state of stress. We note that (1.4) and (1.5) yields the values $M = \nu$, $M = \nu/(1+\nu)$, respectively, for the elasticity problem where ν is Poisson's ratio. Here and henceforth, the expression $a \equiv b$ replaces the sentence: we denote b in terms of a.

Let the relaxation kernels be represented in the form /6/

$$R_i(\tau,\xi) = r_i(\tau,\xi)(\tau-\xi)^{-\beta} + r_i{}^*(\tau,\xi), \quad i = 1, 2, \quad \beta = \text{const} < 1 \tag{1.6}$$

The functions $r_i, r_i{}^*, G, E^*$ are bounded and continuous in τ, ξ and the moduli G, E^* are not degenerate

$$G(\tau) \geqslant \text{const} > 0, \quad E^*(\tau) \geqslant \text{const} > 0 \tag{1.7}$$

Let the actions $T(\mathbf{x},\tau)$, $f_i(\mathbf{x},\tau)$, $F_i(\mathbf{x},\tau)$ satisfy the estimates

$$\|T(\cdot,\tau),\ L_2(\Omega(\tau))\| \leqslant c_T$$
$$\sum_{i=1}^{2}(\|f_i(\cdot,\tau), L_2(\Omega(\tau))\| + \|F_i(\cdot,\tau), L_2(\partial\Omega(\tau))\|) \leqslant c_F, \quad \tau \in [0,t] \tag{1.8}$$

c_T, c_F are positive constants. We will call the functions $\sigma_{ij}{}^*(\mathbf{x},\tau)$, $\varepsilon_{ij}{}^*(\mathbf{x},\tau)$, $u_i{}^*(\mathbf{x},\tau)$, defined for $\tau \in [0,t]$, $\mathbf{x} \in \Omega(\tau)$, satisfying the relationships (1.1)-(1.3) and the estimates

$$\sum_{i,j=1,2}(\|\sigma_{ij}{}^*(\cdot,\tau), L_2(\Omega(\tau))\|) \leqslant c_\sigma,$$
$$\sum_{i,j=1,2}(\|\varepsilon_{ij}{}^*(\cdot,\tau),\ \ L_2(\Omega(\tau))\|) \leqslant c_\varepsilon \tag{1.9}$$

for almost all $\tau \in [0,t]$, the solution of the problem (1.1)-(1.3).

Here and henceforth, we consider the displacement fields that differ by a rigid shift as coincident.

We call A_T (A_F, respectively) the problem (1.1)-(1.3) with zero forces (temperature).

Theorem 1. Under the assumptions made, a single solution $\sigma_{ij}{}^T, \varepsilon_{ij}{}^T, u_i{}^T$ of the problem A_T exists, which can be represented in the form

$$\sigma_{ij}{}^T(\mathbf{x},\tau) = (2G_1)^{-1}(1-\nu_1)\,\sigma_{ij}{}^1(\mathbf{x},\tau), \quad \mathbf{x} \in \Omega(\tau), \quad i,j = 1,2 \tag{1.10}$$

$$\varepsilon_{ij}{}^T(\mathbf{x},\tau) = \sum_{k=1}^{2}\alpha_k{}^T[\varepsilon_{ij}{}^k(\mathbf{x},\cdot)](\tau), \tag{1.11}$$

$$u_i{}^T(\mathbf{x},\tau) = \sum_{k=1}^{2}\alpha_k{}^T[u_i{}^k(\mathbf{x},\cdot)](\tau), \quad \mathbf{x} \in \Omega(\tau), \quad i,j = 1,2$$

$$\alpha_1{}^T \equiv \frac{1-\nu_1}{\nu_2-\nu_1}W^{-1}(\nu_2 I - M) \quad (1 \leftrightarrow 2) \tag{1.12}$$

Here and henceforth the relationship not written down is obtained by a circular permutation of the subscripts indicated in the parentheses; $\sigma_{ij}{}^k$, $\varepsilon_{ij}{}^k$, $u_i{}^k$ $(k = 1, 2)$ are two solutions of the auxiliary elasticity theory problem in the domain $\Omega(\tau)$ that satisfy (1.1)-(1.3) for $f_i = F_i = 0$ and the elasticity law

$$\varepsilon_{ij}^k(\mathbf{x}, \tau) = \varepsilon_{ij}^0(\mathbf{x}, \tau) + \frac{1}{2G_k}(\sigma_{ij}^k(\mathbf{x}, \tau) - \delta_{ij}\nu_k\sigma_{ss}^k(\mathbf{x}, \tau)), \tag{1.13}$$
$$\mathbf{x} \in \Omega(\tau), \quad i, j = 1, 2$$
$$\varepsilon_{ij}^0(\mathbf{x}, \tau) \equiv (I - M)^{-1} W[\varepsilon_{ij}^B(\mathbf{x}, \cdot)](\tau), \quad \mathbf{x} \in \Omega(\tau), \quad i, j = 1, 2 \tag{1.14}$$

The constants G_k, ν_k in (1.10)-(1.13) and the formulas (1.17)-(1.20), (1.22) presented later, are arbitrary, Poisson's ratio ν_k are distinct, and for the plane state of stress ν_k is understood to be the transformed ratio $\nu_k/(1+\nu_k)$.

Theorem 2. We consider a special case of problem A_F in which the bulk forces are zero while loads F_i are selfequilibrated on the inner contours and slits

$$f_i = 0, \quad \int\limits_{\partial\omega_j(\tau)} F_i(\mathbf{x}, \tau)\, dS = 0 \tag{1.15}$$
$$\int\limits_{\gamma_j^+(\tau)} F_i(\mathbf{x}, \tau)\, dS + \int\limits_{\gamma_j^-(\tau)} F_i(\mathbf{x}, \tau)\, dS = 0, \quad j = 1, \ldots, N+J,$$
$$i = 1, 2$$

Under the assumptions made, a single solution σ_{ij}^F, ε_{ij}^F, u_i^F of the problem A_F exists, which can be represented in the form

$$\sigma_{ij}^F(\mathbf{x}, \tau) = \sigma_{ij}^1(\mathbf{x}, \tau), \quad \mathbf{x} \in \Omega(\tau), \quad i, j = 1, 2 \tag{1.16}$$
$$\varepsilon_{ij}^F(\mathbf{x}, \tau) = \sum_{k=1}^{2} \alpha_k^F[\varepsilon_{ij}^k(\mathbf{x}, \cdot)](\tau) \tag{1.17}$$
$$u_i^F(\mathbf{x}, \tau) = \sum_{k=1}^{2} \alpha_k^F[u_i^k(\mathbf{x}, \cdot)](\tau), \quad \mathbf{x} \in \Omega(\tau), \quad i, j = 1, 2$$
$$\alpha_1^F = \frac{2G_1}{\nu_2 - \nu_1} W^{-1}(\nu_2 I - M) \quad (1 \leftrightarrow 2)$$

Here σ_{ij}^k, ε_{ij}^k, u_i^k $(k = 1, 2)$ are two solutions of the auxiliar elasticity theory problem B_F in the domain $\Omega(\tau)$ satisfying (1.1), (1.2) and the elasticity law (1.13) in which $\varepsilon_{ij}^0 = 0$.

Theorem 3. Suppose the domain $\Omega(\tau) = \Omega(0)$, i.e., is fixed. Then the solution σ_{ij}^F, ε_{ij}^F, u_i^F of the problem A_F can be represented in the form

$$\sigma_{ij}^F(\mathbf{x}, \tau) = \beta_1^F[\sigma_{ij}^1(\mathbf{x}, \cdot)](\tau) + \beta_2^F[\sigma_{ij}^2(\mathbf{x}, \cdot)](\tau), \tag{1.18}$$
$$\mathbf{x} \in \Omega(0), \quad i, j = 1, 2$$
$$\beta_1^F \equiv \frac{(1-\nu_1)(1-\nu_2)}{\nu_2 - \nu_1}\left(\frac{\nu_2}{1-\nu_2} I - P\right) \quad (1 \leftrightarrow 2)$$
$$P \equiv (I - M)^{-1} M, \quad MP = P - M$$
$$\varepsilon_{ij}^F(\mathbf{x}, \tau) = \sum_{k=1}^{3} \alpha_k^F[\varepsilon_{ij}^k(\mathbf{x}, \cdot)](\tau), \tag{1.19}$$
$$u_i^F(\mathbf{x}, \tau) = \sum_{k=1}^{3} \alpha_k^F[u_i^k(\mathbf{x}, \cdot)](\tau) \quad \mathbf{x} \in \Omega(0), \quad i, j = 1, 2$$
$$\alpha_1^F = \frac{2G_1(1-\nu_1)}{(\nu_1 - \nu_2)(\nu_1 - \nu_3)} W^{-1}(P(1-\nu_2)(1-\nu_3) - M + \nu_2\nu_3 I) \tag{1.20}$$
$$(1 \to 2 \to 3 \to 1)$$

Here σ_{ij}^k, ε_{ij}^k, u_i^k $(k = 1, 2, 3)$ are three solutions of the auxiliary elasticity theory problem B_F in the domain $\Omega(0)$ satisfying (1.1), (1.2) and the elasticity law (1.13) in which $\varepsilon_{ij}^0 = 0$.

We note that (1.18) has been obtained in /7/ for the elasticity problem with no mass forces present.

Theorem 4. Let $\sigma_{ij}^0(\mathbf{x}, \tau)$ be an arbitrary statically admissible stress field in the domain $\Omega(\tau)$, i.e., $\sigma_{ij}^0(\mathbf{x}, \tau)$ satisfies the identity (1.2) in $\Omega(\tau)$. We introduce the quantities $\varepsilon_{ij}^B(\mathbf{x}, \tau)$, $\varepsilon_{ij}^0(\mathbf{x}, \tau)$ by the formulas

$$\varepsilon_{ij}^B(\mathbf{x}, \tau) = W^{-1}[\sigma_{ij}^0(\mathbf{x}, \cdot)](\tau) - \delta_{ij} W^{-1} M[\sigma_{ss}^0(\mathbf{x}, \cdot)](\tau) \tag{1.21}$$
$$\varepsilon_{ij}^0(\mathbf{x}, \tau) = (I - M)^{-1} W[\varepsilon_{ij}^B(\mathbf{x}, \cdot)](\tau), \quad \mathbf{x} \in \Omega(\tau), \quad i, j = 1, 2$$

Under the assumptions made, a unique solution σ_{ij}^F, ε_{ij}^F, u_i^F of the problem A_F exists, which can be represented in the form

$$\sigma_{ij}^F(\mathbf{x},\tau)=\sigma_{ij}^0(\mathbf{x},\tau)+\frac{1-\nu_1}{2G_1}\sigma_{ij}^1(\mathbf{x},\tau),\quad \mathbf{x}\in\Omega(\tau),\quad i,j=1,2 \tag{1.22}$$

$$\varepsilon_{ij}^F(\mathbf{x},\tau)=\sum_{k=1}^{2}\alpha_k^T[\varepsilon_{ij}^k(\mathbf{x},\cdot)](\tau) \tag{1.23}$$

$$u_i^F(\mathbf{x},\tau)=\sum_{k=1}^{2}\alpha_k^T[u_i^k(\mathbf{x},\cdot)](\tau),\quad \mathbf{x}\in\Omega(\tau),\quad i,j=1,2$$

The operators α_1^T, α_2^T in (1.23) are given by the relationships (1.12) and σ_{ij}^k, ε_{ij}^k, u_i^k are two solutions of the auxiliary elasticity theory problem B in the domain $\Omega(\tau)$ satisfying Eqs.(1.1), (1.2) for $f_i=F_i=0$, and the elasticity law (1.13) in which ε_{ij}^0 are given by (1.21).

2. We use the notation

$$E(\tau)=\{\mathbf{e}\,|\,\mathbf{e}=\{e_{ij}\},e_{ij}=e_{ji},e_{ij}\in L_2(\Omega(\tau)),i,j=1,2\}$$

for the Hilbert space of symmetric tensor fields of second rank with the scalar product

$$(\mathbf{e},\boldsymbol{\varepsilon})_\tau=\int\limits_{\Omega(\tau)}e_{ij}\varepsilon_{ij}\,dx$$

It is known /8/ that $E(\tau)$ decomposes into the direct sum of the subspaces $K(\tau)$ and $L(\tau)$, where $K(\tau)$ is the set of stress fields satisfying the homogeneous equilibrium equations and homogeneous boundary conditions in a generalized sense, and $L(\tau)$ is the set of strain fields generated by an arbitrary displacement field from $W_2^1(\Omega(\tau))$:

$$L(\tau)=\{\boldsymbol{\varepsilon}\,|\,\varepsilon_{ij}=2^{-1}(u_{i,j}+u_{j,i}),u_i\in W_2^1(\Omega(\tau)),i,j=1,2\}$$
$$K(\tau)=\{\boldsymbol{\sigma}\,|\,(\boldsymbol{\sigma},\boldsymbol{\varepsilon})_\tau=0,\quad\forall\boldsymbol{\varepsilon}\in L(\tau)\}$$

Let $K^\infty(\tau)$ denote the subset of $K(\tau)$ consisting of the fields $\boldsymbol{\sigma}\in C^\infty(\Omega(\tau))$, having bounded derivatives of arbitrary order.

It is known /9/ that for any smooth field $\boldsymbol{\sigma}\in K(\tau)$ a smooth Airy function exists in $\Omega(\tau)$

$$\sigma_{11}=\psi_{,22},\quad\sigma_{12}=\sigma_{21}=-\psi_{,12},\quad\sigma_{22}=\psi_{,11} \tag{2.1}$$

The Airy function is determined to the accuracy of an arbitrary linear component and for each contour $\partial\omega_j(\tau)$ and slit $\gamma_j(\tau)$ a polynomial $p_j(\mathbf{x})=a_j+b_jx_1+c_jx_2$ exists such that

$$\psi(\mathbf{x})=p_j(\mathbf{x}),\quad\psi_{,k}(\mathbf{x})=p_{j,k}(\mathbf{x}),\quad\mathbf{x}\in\partial\omega_j(\tau), \tag{2.2}$$
$$j=1,\ldots,N,\quad k=1,2$$

$$\psi^+(\mathbf{x})=\psi^-(\mathbf{x})=p_j(\mathbf{x}),\quad\psi_{,k}^+(\mathbf{x})=\psi_{,k}^-(\mathbf{x})=p_{j,k}(\mathbf{x}), \tag{2.3}$$
$$\mathbf{x}\in\gamma_j(\tau),\quad j=N+1,\ldots,N+J$$

The plus and minus superscripts denote values of the functions on $\gamma^\pm$.

Lemma 1. 1°. The subspace $K(\tau)$ consists of those and only those fields $\boldsymbol{\sigma}$ for which an Airy function $\psi\in W_2^2(\Omega(\tau))$ exists that satisfies (2.1)-(2.3) (we denote the space of all such functions by $\Psi(\tau)$).

2°. The set $K^\infty(\tau)$ is compact in $K(\tau)$ in the metric of the space $E(\tau)$.

Proof. We will prove the necessity of the condition 1°.

Let σ be an arbitrary field from $K(\tau)$. We denote the continuation of σ to zero on R_2 by $\mathbf{s}=\{s_{ij}\}$. For each field $\mathbf{u}\in W_2^1(R_2)$ the following chain of equalities obviously holds:

$$0=(\sigma,\varepsilon(\mathbf{u}^\rho))_\tau=\int\limits_{R_2}s_{ij}\varepsilon_{ij}(\mathbf{u}^\rho)\,dx=\int\limits_{R_2}s_{ij}^\rho\varepsilon_{ij}(\mathbf{u})\,dx \tag{2.4}$$

Here $\mathbf{u}^\rho$, $\mathbf{s}^\rho$ are respectively Sobolev averages of fields $\mathbf{u}$, $\mathbf{s}$ /10/.

Since the functions s_{ij}^ρ are smooth, the existence of the Airy function ψ_ρ for s^ρ follows from (2.4). The field $\mathbf{s}^\rho$ equals zero outside Ω_ρ (ρ is the neighbourhood of the domain Ω_0), consequently the function ψ_ρ agrees with the polynomial outside Ω_ρ. Subtracting this polynomial from ψ_ρ and retaining the previous notation as a result, we obtain that all the functions ψ_ρ equal zero outside Ω_d for $\rho\leqslant d$ and d is fixed. Then $\|\psi_\rho,W_2^2(\Omega_d)\|\sim\|\mathbf{s}^\rho,L_2(\Omega_d)\|$ and from the convergence of $\mathbf{s}^\rho$ to $\mathbf{s}$ in the metric $L_2(\Omega_d)$ it follows that the function ψ_ρ converges in the metric $W_2^2(\Omega_d)$ to a certain function ψ such that $\psi=\partial\psi/\partial n=0$ on $\partial\Omega_d$, the relationships (2.1) hold and the following equality holds:

$$\psi^+(\mathbf{x})=\psi^-(\mathbf{x}),\quad\psi_{,k}^+(\mathbf{x})=\psi_{,k}^-(\mathbf{x}),\quad k=1,2;\quad\mathbf{x}\in\gamma_j^\pm(\tau), \tag{2.5}$$
$$j=N+1,\ldots,N+J$$

Let $\mathbf{u}$ be a smooth field in $\Omega(\tau)$. Using (2.1) we obtain

$$0 \quad (\sigma, \varepsilon(\mathbf{u}))_\tau = \int\limits_{\Omega(\tau)} (\psi_{,22}u_{1,1} - \psi_{,12}u_{1,2} - \psi_{,21}u_{2,1} + \psi_{,11}u_{2,2})\, dx \tag{2.6}$$
$$\int\limits_{\partial\Omega(\tau)} (\psi_{,2}(u_{1,1}n_2 - u_{1,2}n_1) + \psi_{,1}(-u_{2,1}n_2 + u_{2,2}n_1))\, dS = \int\limits_{\partial\Omega(\tau)} \left(-\psi_{,2}\frac{\partial u_1}{\partial S} + \psi_{,1}\frac{\partial u_2}{\partial S}\right) dS$$

Here $\mathbf{n} = (n_1, n_2)$ is the external normal to the boundary $\partial\Omega(\tau)$, and $\partial/\partial S$ is the derivative with respect to the tangent $\boldsymbol{\tau} = (-n_2, n_1)$. Because of the arbitrariness of $\mathbf{u}$ it follows from (2.6) and (2.5) that the derivatives $\psi_{,1}$, $\psi_{,2}$ are constant on the contours $\partial\omega_j(\tau)$ and the slits $\gamma_j(\tau)$.

We express ψ in terms of $b_j \equiv \psi_{,1}$, $c_j \equiv \psi_{,2}$ on $\gamma_j(\tau)$ or $\partial\omega_j(\tau)$:

$$\psi(S) = \psi(S_0) + \int\limits_{S_0}^{S} (\psi_{,1}\, dx_1/dS + \psi_{,2}\, dx_2/dS) = \psi(S_0) - b_j x_1(S_0) - c_j x_2(S_0) + b_j x_1(S) + c_j x_2(S)$$

which proves (2.2) and (2.3).

The sufficiently of condition 1° follows from the disappearance of the right-hand side in (2.6) for the function $\psi \in \Psi(\tau)$.

Now let $\psi \in \Psi(\tau)$. Let $\varphi_j(x)$ be the cutoff functions that equal one near the appropriate contour $\partial\omega_j(\tau)$ or slit $\gamma_j(\tau)$. Then

$$\psi = w + \sum_{j=1}^{N+J} \varphi_j(\mathbf{x})\, p_j(\mathbf{x}), \quad w \in W_2^{02}(\Omega(\tau))$$

It is known that the set of finite functions is compact in the space $W_2^{02}(\Omega(\tau))$ which proves assertion 2° of the lemma because of (2.7).

Corollary. Let $\psi \in \Psi(\tau)$ be the Airy function of the field $\sigma \in K(\tau)$. We continue ψ and σ from $\Omega(\tau)$ into Ω_0 by predefining $\psi(x) = p_j(x)$, $\sigma(\mathbf{x}) = 0$ for $\mathbf{x} \in \omega_j(\tau)$, $j = 1, \ldots, N$; then ψ and σ are connected by relationships (2.1) in Ω_0. We retain the notation $\Psi(\tau)$, $K(\tau)$ as the sets of continued functions. It follows from Lemma 1 that $\Psi(\tau_1) \subset \Psi(\tau_2)$, $K(\tau_1) \subset K(\tau_2)$ for $\tau_1 > \tau_2$.

Lemma 2 /11/. Let Q be any of the operators $W, L, W^{-1}, L^{-1}, M, P, (I - M)^{-1}$ and Ω an arbitrary domain with fixed boundary. Then the linear mapping $g(\mathbf{x}, \zeta) = Q[\varphi(\mathbf{x}, \cdot)](\zeta)$ carries the space $L^\infty(0, \tau; L_2(\Omega))$ over into itself continuously and the estimate with constant $c(\tau)$ dependent on τ is valid

$$\| g; L^\infty(0, \tau; L_2(\Omega)) \| \leqslant c(\tau) \| \varphi; L^\infty(0, \tau; L_2(\Omega)) \|$$

Lemma 3. The solutions $\boldsymbol{\sigma}^k$, $\boldsymbol{\varepsilon}^k$ of the auxiliary elasticity problems B_T, B_F satisfy the estimates (1.9) (with σ_{ij}^*, ε_{ij}^* therein replaced by σ_{ij}^k, ε_{ij}^k, respectively).

The proof follows from the assumptions (1.8) about the nature of the change in the boundary of $\partial\Omega(\tau)$ and the known energy estimates of the solutions of the elasticity problem /8/.

3. We prove Theorems 1-4. We formulate the creep problem in terms of stresses by using the law (1.3) and writing the condition $\boldsymbol{\varepsilon}(\cdot, \tau) \in L(\tau)$ in the form

$$\int\limits_{\Omega(\tau)} (\varepsilon_{ij}^B(\mathbf{x}, \tau) + W^{-1}[\sigma_{ij}(\mathbf{x}, \cdot)](\tau) - \delta_{ij}W^{-1}M[\sigma_{ss}(\mathbf{x}, \cdot)](\tau)) \times s_{ij}(\mathbf{x})\, dx = 0, \quad \forall \mathbf{s} \in K(\tau) \tag{3.1}$$

If a solution exists for the creep problem then the stresses $\boldsymbol{\sigma}^*(\mathbf{x}, \tau)$ satisfy (3.1). The converse is also true: if $\boldsymbol{\sigma}^*$ satisfies the identities (1.2) and (3.1) and the estimate (1.9) for $\tau \in [0, t]$, then the estimate (1.9) is valid (because of Lemma 2) for strain $\boldsymbol{\varepsilon}^*$ determined from (1.13) in terms of $\boldsymbol{\sigma}^*$ and a displacement field $\mathbf{u}^*$ exists that generates the strain $\boldsymbol{\varepsilon}^*$ according to (1.1).

We note that the known conditions of single-valuedness of the displacements and the angle of rotation expressed in terms of stresses during traversal of the inner contours /9/ are natural conditions of the identity (3.1).

We assume that a $\boldsymbol{\sigma}$ exists that satisfies (1.2) and (3.1). We fix τ_1 and substitute the field $\mathbf{s} \in K(\tau_1)$ into (3.1) for $\tau \leqslant \tau_1$ (we denote the Airy function for $\mathbf{s}$ by ψ)

$$\int\limits_{\Omega(\tau_1)} (\varepsilon_{ij}^B(\mathbf{x}, \tau) + W^{-1}[\sigma_{ij}(\mathbf{x}, \cdot)](\tau) - \delta_{ij}W^{-1}M[\sigma_{ss}(\mathbf{x}, \cdot)](\tau)) \times (s_{ij}(\mathbf{x})\, dx = 0, \quad \tau \leqslant \tau_1 \tag{3.2}$$

Since the domain of integration is fixed in (3.2), the operator W can be applied to (3.2). We arrive at the identity

$$\int\limits_{\Omega(\tau_1)} (W[\varepsilon_{ij}^B(\mathbf{x}, \cdot)](\tau) + \sigma_{ij}(\mathbf{x}, \tau) - \delta_{ij}M[\sigma_{ss}(\mathbf{x}, \cdot)](\tau))\, s_{ij}(\mathbf{x})\, dx = 0, \qquad \tau \leqslant \tau_1 \tag{3.3}$$

We substitute the displacements $u_1 = \psi_{,1}$, $u_2 = \psi_{,2}$ into (1.2)

$$\int_{\Omega(\tau_1)} \sigma_{ij}(\mathbf{x}, \tau)\psi_{,ij}(\mathbf{x})\,dx = \int_{\Omega(\tau)} f_i(\mathbf{x}, \tau)\psi_{,i}(\mathbf{x})\,dx + \int_{\partial\Omega(\tau)} F_i(\mathbf{x}, \tau)\psi_{,i}(\mathbf{x})\,dS, \quad \tau \leqslant \tau_1 \tag{3.4}$$

Combining (3.3) and (3.4) and using (2.1), we obtain the identity

$$\int_{\Omega(\tau_1)} (W[\varepsilon_{ij}^B(\mathbf{x}, \cdot)](\tau) s_{ij}(\mathbf{x}) + (I - M)[\sigma_{ss}(\mathbf{x}, \cdot)](\tau)\Delta\psi(\mathbf{x}))\,dx = \int_{\Omega(\tau)} f_i(\mathbf{x}, \tau)\psi_{,i}(\mathbf{x})\,dx + \int_{\partial\Omega(\tau)} F_i(\mathbf{x}, \tau)\psi_{,i}(\mathbf{x})\,dS, \quad \tau \leqslant \tau_1 \tag{3.5}$$

Let us prove the uniqueness of the solution of problem (1.2), (3.1). To do this it is sufficient to show that only a zero solution exists for $\varepsilon_{ij}^B = f_i = F_i = 0$. In this case (3.5) is converted to the form

$$\int_{\Omega(\tau_1)} \sigma_{ss}(\mathbf{x}, \tau)\Delta\psi(\mathbf{x})\,dx = 0, \quad \tau \leqslant \tau_1 \tag{3.6}$$

We put $\tau = \tau_1$ in (3.6) and ψ equal to the Airy function of the field σ and we subtract (3.4) with zero right-hand side from (3.6), we obtain

$$0 = \int_{\Omega(\tau_1)} ((\Delta\psi)^2 - (\psi_{,22}\psi_{,11} - 2(\psi_{,12})^2 + \psi_{,11}\psi_{,22}))\,dx = \int_{\Omega(\tau_1)} \psi_{,ij}\psi_{,ij}\,dx$$

and consequently, $\sigma = 0$, and the uniqueness is proved.

We consider the problem A_T (Theorem 1). Applying the operator $(I - M)^{-1}$ to (3.5), we convert (3.5) to the form

$$\int_{\Omega(\tau_1)} (\varepsilon_{ij}^0(\mathbf{x}, \tau) s_{ij}(\mathbf{x}) + \sigma_{ss}(\mathbf{x}, \tau)\Delta\psi(\mathbf{x}))\,dx = 0, \quad \tau \leqslant \tau_1 \tag{3.7}$$

Here ε_{ij}^0 is determined from (1.14). It is similarly confirmed that the solutions σ_{ij}^k of the problem B_T satisfy the identity

$$\int_{\Omega(\tau_1)} \left(\varepsilon_{ij}^0(\mathbf{x}, \tau) s_{ij}(\mathbf{x}) + \frac{1 - \nu_k}{2G_k}\sigma_{ss}^k(\mathbf{x}, \tau)\Delta\psi(\mathbf{x})\right)dx = 0, \quad k = 1, 2 \tag{3.8}$$

From (3.8) we obtain the equality

$$\frac{1 - \nu_1}{2G_1}\sigma_{ij}^1 = \frac{1 - \nu_2}{2G_2}\sigma_{ij}^2 \tag{3.9}$$

Comparing (3.7) and (3.8) we note that if the σ_{ij}^T is determined by (1.10), then the field σ_{ij}^T will satisfy the identities (1.2), (3.7) and, because of the reversibility of all calculations, the identity (3.1), and hence the validity of (1.10), follows when the uniqueness of the solution and Lemma 3 are taken into account.

We consider the problem A_F under the conditions of Theorem 2. It follows from (2.2), (2.3) and (1.15) that all the components in (3.5) that contain effects vanish and the application of the operator $(I - M)^{-1}$ to (3.5) results in the identity (3.7) in which $\varepsilon_{ij}^0 = 0$ while the corresponding transformation of the problem B_F results in (3.8) in which $\varepsilon_{ij}^0 = 0$, which proves (1.16).

We examine the problem A_F under the conditions of Theorem 3, then ε_{ij}^B vanishes in (3.5) and $\Omega(\tau) = \Omega(\tau_1) = \Omega(0)$. The corresponding identity for the solutions of the problem B_F appears as

$$\int_{\Omega(0)} (1 - \nu_k)\sigma_{ss}^k(\mathbf{x}, \tau)\Delta\psi(\mathbf{x})\,dx = \int_{\Omega(0)} f_i(\mathbf{x}, \tau)\psi_{,i}(\mathbf{x})\,dx + \int_{\partial\Omega(0)} F_i(\mathbf{x}, \tau)\psi_{,i}(\mathbf{x})\,dS, \quad k = 1, 2, 3 \tag{3.10}$$

We shall seek the solution σ_{ij}^F of the identities (3.5), (1.2) in the form (1.18). Substitution of (1.18) into (1.2) and (3.5), taking (3.10) into account yields an equation for finding β_1^F, β_2^F

$$\beta_1^F + \beta_2^F = I, \quad (I - M)((1 - \nu_1)^{-1}\beta_1^F + (1 - \nu_2)^{-1}\beta_2^F) = I \tag{3.11}$$

Solving (3.11), we obtain the values of the operators β_1^F, β_2^F presented in (1.18). Applying (1.8) to the solution σ^3, we obtain the representation

$$\sigma_{ij}^3 = \frac{(1 - \nu_1)(1 - \nu_2)}{(\nu_2 - \nu_1)}\left(\left(\frac{\nu_2}{1 - \nu_2} - \frac{\nu_3}{1 - \nu_3}\right)\sigma_{ij}^1 + \left(\frac{\nu_3}{1 - \nu_3} - \frac{\nu_1}{1 - \nu_1}\right)\sigma_{ij}^2\right), \quad i, j = 1, 2 \tag{3.12}$$

We examine the problem A_F under the conditions of Theorem 4. The solution σ^F of the identities (1.2), (3.1) will be sought in the form $\sigma_{ij}^F = \sigma_{ij}^0 + \mu_{ij}$, where $\mu \in K(\tau)$. Then μ_{ij} satisfy the identity (3.1) in which σ_{ij} are replaced by μ_{ij}, while ε_{ij}^B are determined by (1.21). Therefore, the problem of finding μ would agree (apart from replacing (1.21) by (1.14), (1.4) or (1.5)) with the problem of determining the solution σ_{ij}^T of the problem A_T, which proves (1.22).

We will prove representation (1.11). Substituting (1.10) into (1.3), we obtain

$$\varepsilon_{ij}^F(\mathbf{x},\tau) = \varepsilon_{ij}^B(\mathbf{x},\tau) + \frac{1-\nu_1}{2G_1}(W^{-1}[\sigma_{ij}^1(\mathbf{x},\cdot)](\tau) - \delta_{ij}W^{-1}M[\sigma_{ss}^1(\mathbf{x},\cdot)](\tau)), \quad \mathbf{x}\in\Omega(\tau_1), \quad \tau\leqslant\tau_1 \tag{3.13}$$

We introduce the linear operator

$$a_{ij}(\mu,Q,D)(\mathbf{x},\tau) = \mu_{ij}(\mathbf{x},\tau) + Q[\sigma_{ij}^1(\mathbf{x},\cdot)](\tau) - \delta_{ij}D[\sigma_{ss}^1(\mathbf{x},\cdot)](\tau), \quad i,j=1,2 \tag{3.14}$$

Here Q, D are arbitrary Volterra operators. It follows from (3.13) and the definition (3.14) that

$$\varepsilon_{ij}^F = a_{ij}\left(\varepsilon^B, \frac{1-\nu_1}{2G_1}W^{-1}, \frac{1-\nu_1}{2G_1}W^{-1}M\right) \tag{3.15}$$

Comparing (3.14) with (1.13) and utilizing (3.9), we can verify the equalities

$$\begin{aligned}\varepsilon_{ij}^1 &= a_{ij}\left(\varepsilon^0, \frac{1}{2G_1}I, \frac{\nu_1}{2G_1}I\right),\\ \varepsilon_{ij}^2 &= a_{ij}\left(\varepsilon^0, \frac{1-\nu_1}{2G_1(1-\nu_2)}I, \frac{(1-\nu_1)\nu_2}{2G_1(1-\nu_2)}I\right)\end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into (1.11), we obtain the relation

$$a_{ij}\left(\varepsilon^B, \frac{1-\nu_1}{2G_1}W^{-1}, \frac{1-\nu_1}{2G_1}W^{-1}M\right) = a_{ij}\left((\alpha_1^T+\alpha_2^T)[\varepsilon^0], \frac{1}{2G_1}\left(\alpha_1^T + \frac{1-\nu_1}{1-\nu_2}\alpha_2^T\right), \frac{1}{2G_1}\left(\nu_1\alpha_1^T + \frac{(1-\nu_1)\nu_2}{1-\nu_2}\alpha_2^T\right)\right)$$

For this to be true when taking account of relation (1.14) α_1^T, α_2^T must satisfy the equations

$$\begin{aligned}&\alpha_1^T + \alpha_2^T = W^{-1}(I-M)\\ &\alpha_1^T + \frac{1-\nu_1}{1-\nu_2}\alpha_2^T = (1-\nu_1)W^{-1}, \quad \nu_1\alpha_1^T + \frac{(1-\nu_1)\nu_2}{1-\nu_2}\alpha_2^T = (1-\nu_1)W^{-1}M\end{aligned} \tag{3.17}$$

System (3.17) has the unique solution (1.12).

Formulas (1.17) and (1.23) are proved similarly.

It remains to give relationships (1.19) and (1.20) a foundation. Substituting (1.18) into (1.13) we obtain

$$\varepsilon_{ij}^F(\mathbf{x},\tau) = \sum_{k=1}^{2}(W^{-1}\beta_k^F[\sigma_{ij}^k(\mathbf{x},\cdot)](\tau) - \delta_{ij}W^{-1}M\beta_k^F[\sigma_{ss}^k(\mathbf{x},\cdot)](\tau)), \quad i,j=1,2 \tag{3.18}$$

We introduce the linear operator

$$a_{ij}(Q_1,Q_2,D_1,D_2)(\mathbf{x},\tau) = \sum_{k=1}^{2}(Q_k[\sigma_{ij}^k(\mathbf{x},\cdot)](\tau) - \delta_{ij}D_k[\sigma_{ss}^k(\mathbf{x},\cdot)](\tau)) \tag{3.19}$$

From (3.18) and (3.19) we have the equality

$$\varepsilon_{ij}^F(\mathbf{x},\tau) = a_{ij}(W^{-1}\beta_1^F, W^{-1}\beta_2^F, W^{-1}M\beta_1^F, W^{-1}M\beta_2^F)(\mathbf{x},\tau), \quad i,j=1,2 \tag{3.20}$$

Comparing (3.19) with (3.13), we obtain

$$\varepsilon_{ij}^1 = a_{ij}\left(\frac{1}{2G_1}I, 0, \frac{\nu_1}{2G_1}I, 0\right), \quad \varepsilon_{ij}^2 = a_{ij}\left(0, \frac{1}{2G_2}I, 0, \frac{\nu_2}{2G_2}I\right) \tag{3.21}$$

We express the strains ε_{ij}^3 in terms of the stresses σ_{ij}^1, σ_{ij}^2 from (1.13), (3.12)

$$\varepsilon_{ij}^3 = \frac{(1-\nu_1)(1-\nu_2)}{2G_3(\nu_2-\nu_1)(1-\nu_3)}a_{ij}\times\left(\frac{\nu_2-\nu_3}{1-\nu_2}I, \frac{\nu_3-\nu_1}{1-\nu_1}I, \frac{(\nu_2-\nu_3)\nu_3}{1-\nu_2}I, \frac{(\nu_3-\nu_1)\nu_3}{1-\nu_1}I\right) \tag{3.22}$$

Substituting (3.20)-(3.22) into (1.19), we obtain a system of four equations in α_1^F, α_2^F, α_3^F

which has a unique solution achievable by (1.20).

It follows from Theorems 1-4 that in all the problems considered the sets of creep problem solutions (including the solutions of the elasticity problems as special cases) are lineals of finite dimensionality for all possible values of the rheological characteristics. Consequently, the available arbitrariness in selecting the constants G_k, ν_k of the auxiliary elasticity problems essentially denotes the possibility of selecting different bases in this lineal.

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A METHOD FOR THE AUTOMATIC EXTINCTION OF DIRECTIONAL FORCES BY MEANS OF BALL SELFBALANCERS*

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We consider the possible extinction of a directional harmonic force by means of two like selfbalacing systems (SBS) leading to rotation in two opposite directions with a frequency equal to the frequency of the acting force. A method of extinguishing circulating forces caused by rotor imperfections by means of ball SBS was described in /1/. The action of directional forces, e.g., forces due to the operation of crank-and-rod mechanisms, is usually extinguished by means of a system of two constant unbalancers rotating in opposite directions. The latter have poor efficiency, however, if the amplitude or direction of the acting force can vary in time. In this case it is best to use a system of two unbalancers, whose values vary in accordance with the variation of the external disturbing force.

The dynamic characteristics of our theoretical model (Fig.1) will be assumed to be the same in all directions at the location of the selfbalancers and to be given as an impedance ξ_c. Let a directional harmonic force $F = 2D\omega^2 \cos(\omega t + \eta_0)$ act on the system at an angle φ_0.

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